8 Jul 26

Last time, we introduced simplicial sets:

$$F_\bullet : \Delta^{op} \longrightarrow \text{Set} \; ; \quad F_\bullet([n]) = \underbrace{F_n}_{\text{set of } n\text{-simplices}} \in \text{Set}$$

Motivation for a simplicial set:

Ex X topological space, get simplicial set

$$F_\bullet := \text{Sing}_\bullet(X)$$

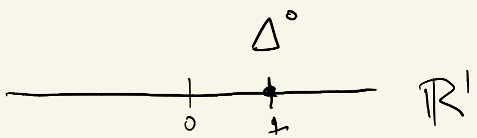
$$\mathrm{sing}_n(X) = C^0(\Delta^n, X)$$

continuous maps from Δ^n (geometric n-simplex) into X.

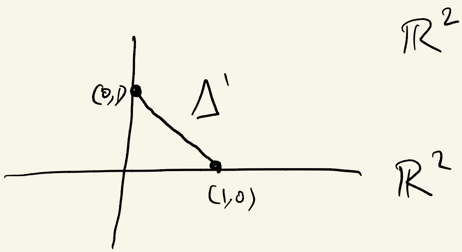
$$\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \}$$

$$\subseteq \mathbb{R}^{n+1}$$

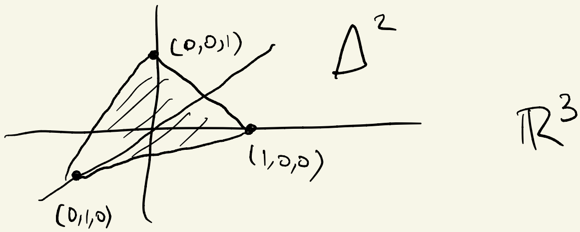
Ex



Ex



Ex



$$f : [m] \longrightarrow [n]$$

$$f_* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$ linear extension

$$e_i \longmapsto e_{f(i)}$$

<u>Exercise</u> This sends Δ^m to Δ^n.

$$f_*(\Delta^m) \subseteq \Delta^n$$

Get $f^* : \mathrm{Sing}_n(X) \longrightarrow \mathrm{Sing}_m(X)$

$$f : [m] \longrightarrow [n]$$ contravariance

$$f^*(\varphi) = \varphi \circ f_*$$

$$\varphi : \Delta^n \to X \longmapsto \varphi \circ f_* : \Delta^m \to X$$

<u>What's the point?</u>

$\mathrm{Sing}_\bullet(X)$ remembers everything we need to know about the homotopy theory of X.

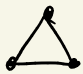
Roughly, $\pi_n(X, x_0)$ nth homotopy group of X, consisting of homotopy classes of maps $\gamma : (S^n, s_0) \to (X, x_0)$

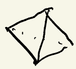
$\gamma(s_0) = x_0$

In simplicial sets

$\partial\Delta^{n+1}$ plays the role of S^n.

<u>Ex</u> $\partial\Delta^0 = \emptyset$, $\partial\Delta^1 = \cdot \quad \cdot = S^0$

$\partial\Delta^2 =$  $\cong S^1$

$\partial\Delta^3 =$  $\cong S^3$

Replace

$\gamma \in \pi_n(X)$ w/ maps $\tilde{\gamma} : \partial\Delta^{n+1} \to \mathrm{Sing}_0(X)$

considered up to homotopy of simplicial maps.

<u>Slogan</u> For a homotopy theorist, simplicial sets are the same as spaces.

For us, we want to use some homotopy theory because it is useful for understanding obstructions:

$$O_{n+1}[L] = \hbar^{-n-1}\left(Q\tilde{I} + \tfrac{1}{2}\{\tilde{I},\tilde{I}\}_L + \hbar\Delta_L\tilde{I}\right)$$

It turns out that the vanishing of $O_{n+1}[L]$ (i.e., finding a solution to the QME_L to order $\hbar^{n+1}$.) is computed cohomologically just in terms of the classical theory.

Recall Classical BV theory

- $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ $\mathbb{Z}$-graded v.b./ spacetime M

- -1-shifted symplectic str

$$\langle -,- \rangle_{loc} : E \otimes E \longrightarrow Dens(M)$$

- Action $S \in \mathcal{O}(\mathcal{E}_c(M))$ cohomological deg 0 satisfying

$$\{S, S\} = 0 \qquad \text{(ME}$$

at least quadratic:

$$S(e) = \langle e, Qe \rangle + I(e)$$

$$I \in \mathcal{O}^+_{loc}(\mathcal{E})$$

$$\{S, S\} = 0$$

$$\Longleftrightarrow \quad QI + \tfrac{1}{2}\{I, I\} = 0$$

(version of the Maurer–Cartan eqn: $d\omega + \frac{1}{2}[\omega, \omega] = 0$, ω is the MC 1-form on a Lie group.)

The MC eqn says that $Q + \{I, -\}$ is also a square-zero differential:

$$\mathcal{O}_{loc}(\mathcal{E}) \longrightarrow \mathcal{O}_{loc}(\mathcal{E})$$
$$F \longmapsto QF + \{I, F\}$$

CME says $(Q + \{I, -\})^2 = 0$.

<u>Think</u> I 'deforms' the free theory Q by deforming the differential:

$$Q \rightsquigarrow Q + \{I, -\}$$

Similar to how $\hbar\Delta$ deforms the classical theory into the quantum.

Pick a gauge fixing operator Q^{GF}.

$$\rightsquigarrow \Delta_L$$

$$\rightsquigarrow \{-,-\}_L$$

$$\rightsquigarrow P(\varepsilon, L)$$

<u>Def</u> A <u>quantization</u> of the BV theory $(\mathcal{E}, I)$ is a family

$$I^q[L] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$$

satisfying RG flow & the QME_L

s.t.

$$I = I^q[L] \bmod \hbar \text{ as } L \to 0.$$
$$= \lim_{L \to 0} I^q[L] \bmod \hbar.$$

Discarding all quantum corrections, we get I back.

Let $\mathcal{T}^{(n)}(\mathcal{E}, I)$ denote the set of quantizations defined modulo $\hbar^{n+1}$.

$\mathcal{T}^{(\infty)}(\mathcal{E}, I)$ quantizations to all orders.

Version of Costello's obstruction theorem as stated in Costello & Gwilliam.

We can extend $\mathcal{T}^{(n)}(\mathcal{E}, I)$ & $\mathcal{T}^{(\infty)}(\mathcal{E}, I)$ to simplicial sets, i.e., spaces:

<u>Thm 7.5.1</u> [Costello–Gwilliam vol 2]

There is a homotopy fiber diagram of simplicial sets:

$$\begin{tikzcd}
\mathcal{T}^{(n+1)}(\mathcal{E}, I) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{T}^{(n)}(\mathcal{E}, I) \arrow[r, "O"] & DK\left(\mathcal{O}_{loc}(\mathcal{E})[1],\ Q + \{I, -\}\right)
\end{tikzcd}$$

obstruction map!

$I^1 \in \mathcal{T}^{(n)}(\mathcal{E}, I)$, then there is an obstruction $O(I^q) \in \mathcal{O}_{loc}(\mathcal{E})$.

$O(I^q)$ is closed wrt $Q + \{I, -\}$ & of deg 1:

$$Q\, O(I^q) + \{I, O(I^q)\} = 0$$

We can lift I^1 to an elt in $\mathcal{T}^{(n+1)}(\mathcal{E}, I)$ iff $O(I^q)$ is exact.

The cohomology groups of the complex:

$$\left(\mathcal{O}_{loc}(\mathcal{E}),\ Q + \{I, -\}\right)$$

KEY

depend only on the classical theory.

So, if we can show they vanish, quantization is unobstructed. In fact, just need to check $H^1 = 0$.

Next time We'll check this for Chern–Simons theory.

We won't prove this, we'll just use it.

Hence, we don't really need to extend $\widetilde{\tau}^{(n)}$ to a simplicial set.

Nevertheless, here's roughly how you do it.

Idea Tensor w/ $\Omega^\bullet(\Delta^n)$, change the differential Q to $Q + d_{dR}$.

Just as we defined f^* for $f_*: \Delta^m \to \Delta^n$ from $f: [m] \to [n]$, we can pullback differential forms.

Def The n-simplices of gauge-fixing operators is the set

$$GF_n = \{ Q^{GF} : \mathcal{E} \otimes \Omega^\bullet(\Delta^n) \longrightarrow \mathcal{E} \otimes \Omega^\bullet(\Delta^n) \}$$

satisfying:

$$Q^{GF} \quad \Omega^\cdot(\Delta^n)\text{-linear, } \deg -1,$$

symmetric wrt $\langle -, - \rangle$,

&

$$D = [Q^{GF}, Q + d_{dR}]$$

is a generalized Laplacian:

$$\sigma(D)_\xi = -|\xi|^2_{\tilde{g}}$$

$\tilde{g}$ is a smooth family of Riemannian metrics on M depending smoothly on Δ^n.

Pullback along

$$f : [m] \to [n]$$
$$f_* : \Delta^m \to \Delta^n$$

makes $GF_\bullet$ into a simplicial set.

Def $\mathcal{T}^{(n)}(\mathcal{E}, I)[k]$ ~ the k-simplices of quantizations up to order n are given by:

- $Q^{GF} \in GF_k = GF[k]$

- A quantization

$$I^q[L] \in \mathcal{O}^+(\mathcal{E})[[\hbar]] \otimes \Omega^\bullet(\Delta^k)$$

satisfying RG flow &

$$(Q + d_{dR}) I^q[L] + \{I^q[L], I^q[L]\}_L + \hbar \Delta_L I^q[L] = 0$$

Again, pullback makes $\mathcal{T}^{(n)}(\mathcal{E}, I)$ into a simplicial set.

For the thm: C be any chain cpx.:

$$C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots$$

$$DK(C)_n = \begin{array}{l}\text{closed deg 0 elts} \\ \text{of } C \otimes \Omega^\bullet(\Delta^n).\end{array}$$

Dold-Kan correspondence

simplicial abelian groups are the same as chain complexes bounded to one side.

13 Jul 2026

Today's goal

Show that Chern-Simons theory admits a BV quantization in the sense of Costello.

To do this, we will argue that the obstruction O_{n+1} to quantizing at order $\hbar^{n+1}$ vanishes.

Recall BV thy of classical CS thy on an oriented 3-mfd M is given by:

$\mathfrak{g} = \mathrm{Lie}(G)$

$$\underline{0}$$

$$\Omega^1(M) \otimes \mathfrak{g}$$

fields

$= \mathfrak{g}$-valued 1-forms A

$=$ connections $d + A$

flat

one-form

we're perturbing around the flat conn on the trivial bundle

$$\mathcal{E}_{cs}: \quad \underset{\text{ghosts}}{\overset{-1}{\Omega^0(M)\otimes\mathfrak{g}}} \xrightarrow{d} \underset{\text{fields}}{\overset{0}{\Omega^1(M)\otimes\mathfrak{g}}} \xrightarrow{d} \underset{\text{antifields}}{\overset{1}{\Omega^2(M)\otimes\mathfrak{g}}} \xrightarrow{d} \underset{\text{antighosts}}{\overset{2}{\Omega^3(M)\otimes\mathfrak{g}}}$$

$$\mathcal{E}_{cs} = \Omega^\bullet(M)\otimes\mathfrak{g}[1]$$

κ invariant bilinear pairing on $\mathfrak{g}$. Ex Killing form.

$$S_{cs}(e) = \int_M \frac{1}{2}\kappa(e, de) + \frac{1}{6}\kappa(e,[e,e])$$

only 3-form part contributes

Note how classical CS is determined by M, $\langle -,- \rangle$, d, $[-,-]$.

$$= \frac{1}{2}\langle e, de\rangle + \frac{1}{6}\langle e,[e,e]\rangle$$

d & $[-,-]$ make $\Omega^\bullet(M)\otimes\mathfrak{g}$ into a dg Lie algebra.

$\mathcal{E}_{cs} = \Omega^\bullet(M) \otimes \mathfrak{g}[1]$

$\mathcal{E}_{cs}[-1] = \Omega^\bullet(M) \otimes \mathfrak{g}$ dg Lie alg

In general, $\mathcal{E}[-1]$ is a dg Lie algebra or an 'L_∞-algebra':

have higher order brackets:

$$[-,-,-], \quad [-,-,-,-], \quad \ldots$$

satisfying a generalization of Jacobi.

<u>Q</u> Why shift by 1?

<u>A</u> ① A Lie algebra has $[-,-]$ of degree 0.

This is true for $\Omega^\bullet(M) \otimes \mathfrak{g}$, but not $\Omega^\bullet(M) \otimes \mathfrak{g}[1]$.

② In derived deformation theory, the tangent complex turns out to always be an L_∞-algebra, after we shift up by 1.

Facts from Costello

Fact 1 Obstructions in CS theory live in $H^1(M, \underbrace{H^3(\mathfrak{g})}_{\substack{\text{vector space,} \\ \text{3rd Lie alg} \\ \text{cohomology of} \\ \mathfrak{g}.}})$.

Fact 2 Deformations of CS thy live in $H^0(M, H^3(\mathfrak{g}))$.

Last time

$$\begin{array}{cc} \tau^{(n)}(\mathcal{E}_{CS}) & \text{s.Set} \\ \downarrow & \\ G\tilde{F} & \text{s.Set} \end{array}$$

Natural choice of GF

$$GF = \mathrm{Met}(M)$$ space of Riemannian metrics

$$g \mapsto Q^{GF} = d^*$$ formal adjoint wrt g.

$$\begin{array}{c} \mathcal{T}^{(n)}(\mathcal{E}_{cs}) \\ \downarrow \\ \mathrm{Met}(M) \end{array}$$

$$CS^n(M) = \Gamma\left(\mathrm{Met}(M), \mathcal{T}^{(n)}(\mathcal{E}_{cs})\right)$$

Similarly $CS^\infty(M)$.

A point in $CS^\infty(M)$ is a universal quantization of CS theory: it gives a quantization for every metric.

Rem This varies in a homotopically trivial way as the metric varies.

Recall obstructions live in

$$\left(\mathcal{O}_{loc}(\mathcal{E}_{cs}),\ \underset{\substack{\| \\ d}}{Q} + \{I_{cs}, -\}\right)$$

They are closed, degree 1 elts of this cpx.

Magic

$$\mathcal{O}_{loc}(\mathcal{E}_{cs}) \underset{q\text{-iso}}{\simeq} \mathrm{Dens}_M \otimes_{D_M} C^\bullet_{red}(\mathfrak{g}) \otimes C^\infty_M$$

Lemma 6.6.2 in Costello + the def'n of $C^\bullet_{red}$ adapted to CS thy.

Lem 6.6.1

$$\mathcal{O}_{loc}(\mathcal{E}) \cong \overbrace{\mathrm{Dens}_M \otimes_{D_M} \overbrace{\underbrace{\mathcal{O}(J(E))}_{\substack{\text{functionals} \\ \text{depending only} \\ \text{on the value} \\ \text{of } e \text{ \& its} \\ \text{derivatives at a pt.}}}}^{\text{Lagrangians}}}^{\text{Lagrangian density}}$$

$$\ni \quad \int_M D_1 e\, D_2 e \cdots D_n e\, \Omega$$

<u>Thm</u> Assume $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = H^4(\mathfrak{g}) = 0$.
(Holds for all $\mathfrak{g}$ semisimple, e.g. $\mathfrak{g} = \mathfrak{su}(2)$). Let M be a connected oriented 3-manifold. Then there is a isomorphism of simp. sets

$$CS^\infty(M) \cong \underbrace{\hbar H^3(\mathfrak{g})[[\hbar]]}_{\text{constant s. set.}}$$

$$\hbar H^3(\mathfrak{g})[[\hbar]]_K = \hbar H^3(\mathfrak{g})[[\hbar]]$$

In particular, CS theory has a quantization.

<u>Sketch of pf</u> By magic, we know obstruction lives in

$$\mathrm{Dens}_M \otimes C^\bullet_{red}(\mathfrak{g}) \otimes C^\infty_M$$

This implies (why?) that the obstruction lives in $H^1(M, H^3(\mathfrak{g}))$.

Note that $\mathcal{E}_{CS}$ form a sheaf on M.

In fact, GF also form a sheaf, & so do theories $\tilde{L}^{(n)}(\mathcal{E})$.

Taking obstructions is a sheaf map:

$$\begin{tikzcd} CS^n(M) \arrow[r, "\text{res to } U"] \arrow[d, "O_{n+1}"'] & CS^n(U) \arrow[d, "O_{n+1}"] & U \subseteq M \text{ open} \\ H^1(M, H^3(\mathfrak{g})) \arrow[r, "\text{res to } U"] & H^1(U, H^3(\mathfrak{g})) & \end{tikzcd}$$

By induction, assume for all 3-mfds M, $\alpha_{k-1} \in CS^{k-1}(M)$ has a lift to order $\alpha_k \in C^k(M)$.

WTS $O_{k+1}(\alpha_k) = O(\alpha_k) = 0$.

We'll do this by restricting to well-chosen opens.

To choose these opens, note

$$H^1(M, H^3(\mathfrak{g})) \cong \mathrm{Hom}(H_1(M,\mathbb{R}), H^3(\mathfrak{g}))$$

since $Ext^1 = 0$ for vector spaces.

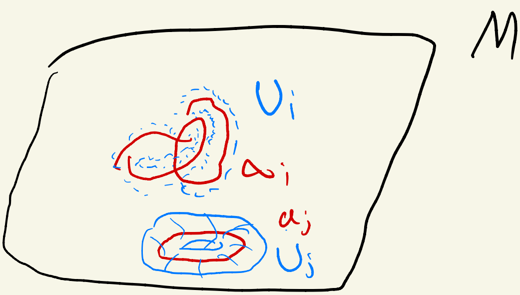
pick a basis $\{\alpha_i\} \subseteq H_1(M, \mathbb{R})$.

pick embedded circles

$$S^1 \xhookrightarrow{\alpha_i} M$$

representing each homology class.

Enlarge these embedded circles to tubular nbhds:



If we can show

$$\mathcal{O}(\alpha_k|_{U_i}) \in H^1(U_i, \mathcal{H}^3(\mathfrak{g}))$$

vanishes for all i, then $\mathcal{O}(\alpha_k) = 0$ on M.

Pick an embedding $U_i \hookrightarrow \mathbb{R}^3$.

By inductive hypothesis:

$$\begin{tikzcd}
CS^k(\mathbb{R}^3) \arrow[r, "\text{res } U_i"'] \arrow[d] & CS^k(U_i) \arrow[d] \\
C^{k-1}(\mathbb{R}^3) \arrow[r, "\text{res } U_i"'] & CS^{k-1}(U_i)
\end{tikzcd}$$

Because deformations are given by

$$H^0(\mathbb{R}^3, \mathcal{H}^1(\mathfrak{g})) \cong H^0(U_i, \mathcal{H}^3(\mathfrak{g}))$$

both $CS^k(\mathbb{R}^3)$ & $CS^k(U_i)$ are torsors (aka principal bundles) for this group.

This implies restriction gives a bijection:

$$CS^\kappa(\mathbb{R}^3) \xrightarrow{\mathrm{res}_{U_i}} CS^\kappa(U_i).$$

Hence $\mathcal{O}(\alpha_\kappa|_{U_i})$ is the image of a class in

$$H^1(\mathbb{R}^3, H^3(\mathfrak{g})) = 0.$$

$$\Rightarrow \mathcal{O}(\alpha_\kappa) = 0$$

All obstructions vanish; $CS^\kappa(M)$ is a torsor of $H^0(M, H^3(\mathfrak{g}))$ $\cong H^3(\mathfrak{g})$. $\blacksquare$

Overview

15 Jul 26

- Perturbed Gaussian integrals & Feynman diagrams

- RG flow:

$$W(P, I) = \hbar \log\left(e^{\hbar \partial_P} e^{I/\hbar}\right)$$
$$= \sum_{\substack{\text{stable} \\ \text{conn } \gamma}} \frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma)|} w_\gamma(P, I)$$

– Scalar thy

$$P(\varepsilon, L) = \int_\varepsilon^L K_\ell \, d\ell$$

– BV thy

$$P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_\ell \, d\ell$$

Def [scalar QFT]

Thm $\Upsilon^{(n+1)} \longrightarrow \Upsilon^{(n)}$ $\mathcal{O}_{loc}$ ~ principal bundle.

BV thy

Def [classical BV thy]

Construction $T^*[-1](V/\!/\mathfrak{g})$

Ex CS thy

$$\langle e, de\rangle + \langle e, [e,e]\rangle$$

Ex YM thy

$$S = \langle F, F\rangle + \langle d_A c, A^\vee\rangle + \langle [c,c], c^\vee\rangle$$

QME

$$QI + \frac{1}{2}\{I,I\}_L + \hbar\Delta_L I = 0$$

Lem RG flow compat w/ QME.

Def sSet GF & $\mathcal{T}^{(n)}$

Main thm

$$\begin{tikzcd}
\mathcal{T}^{(n+1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{T}^{(n)} \arrow[r, "O_{n+1}"] & DK(\mathcal{O}_{loc}(\mathcal{E})[1], Q+\{I,-\})
\end{tikzcd}$$

Ex CS thy